Classifying weighted graphs up to Clifford group equivalence

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20 April 2023

Joint work with Simeon Ball

Overview

$n \in \mathbb{N}, p$ prime

Let Γ be a graph on the vertices $\{1, \ldots, n\}$ with adjacency matrix

$$A = (a_{ij})_{1 \leq i,j \leq n}$$

such that:

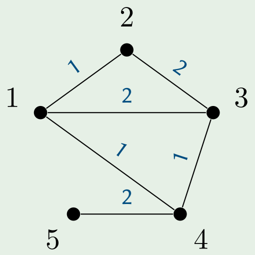
$$a_{ij} \in \mathbb{F}_p \qquad \text{(weighted)}$$

$$a_{ij} = a_{ji} \qquad \text{(undirected)}$$

$$a_{ii} = 0 \qquad \text{(no loops)}$$

$n \in \mathbb{N}, p$ prime

Example ($n = 5$, $p = 3$)



$$A = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

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Definition

For every vertex k and $c \in \mathbb{F}_p \setminus \{0\}$, define

- $f_k(\Gamma) = \Gamma'$ where
$$a'_{ij} = a_{ij} + a_{ik}a_{jk} \quad (i \neq j)$$
- $g_{k,c}(\Gamma) = \Gamma'$ where
$$a'_{ij} = \begin{cases} c \cdot a_{ij} & \text{if } i = k \text{ or } j = k \\ a_{ij} & \text{else} \end{cases}$$

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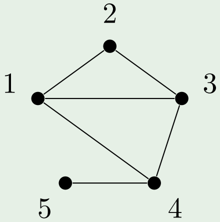
Clifford group equivalence

$\Gamma \sim_C \Gamma'$ if there is a sequence of f_k and $g_{k,c}$ that converts Γ into Γ'.

Example ($p = 2$)



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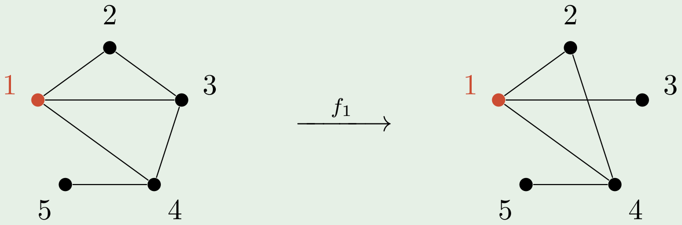

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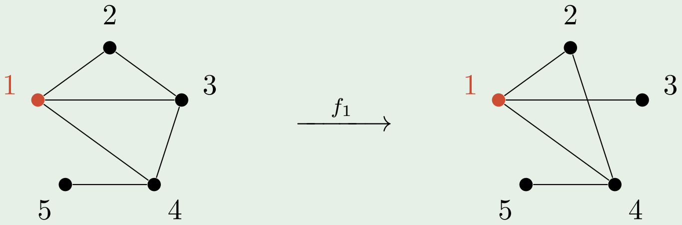
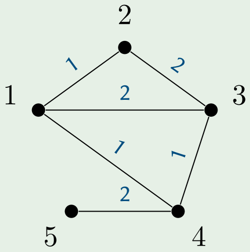


$\rightarrow$ **local complementation**

Example ($p = 3$)



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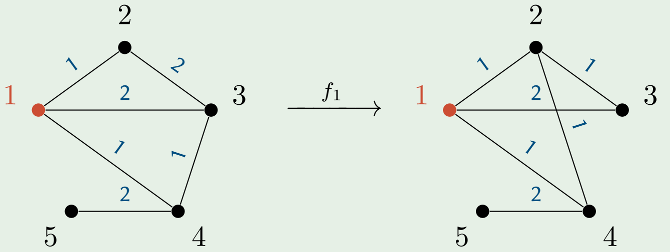
Example ($p = 3$)



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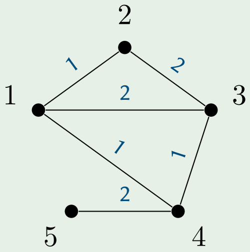

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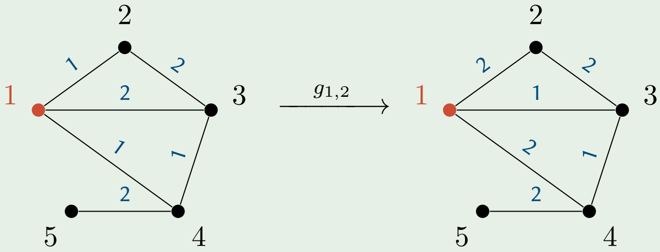

Question

How many equivalence classes does $\sim_C$ have?

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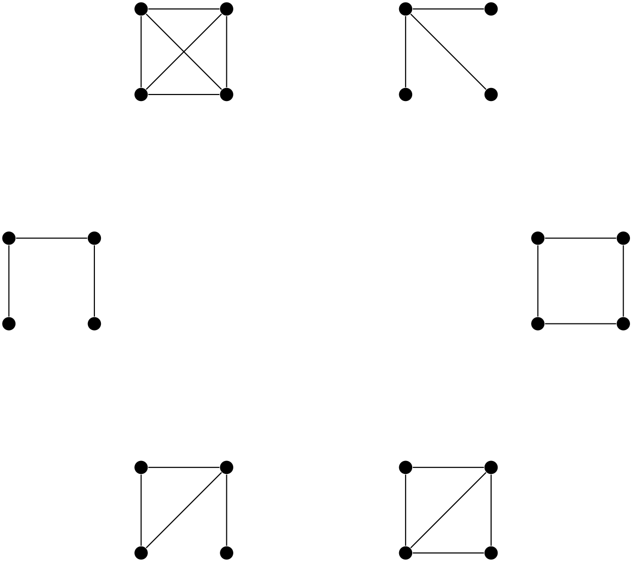
$\rightarrow$ w.l.o.g. Γ connected

Graphs



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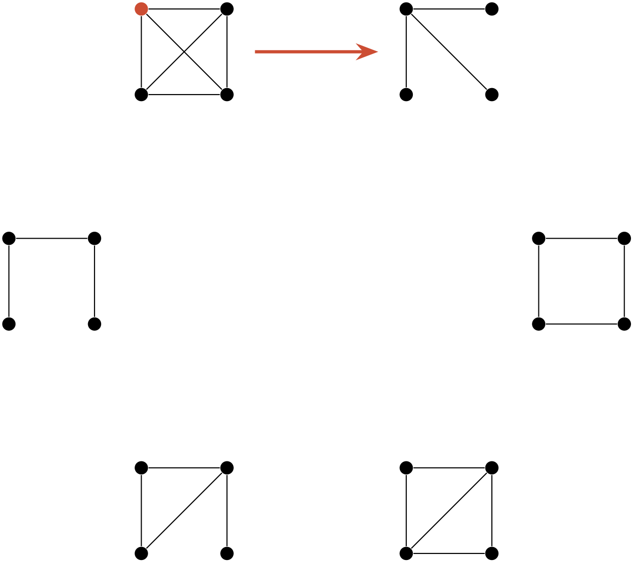

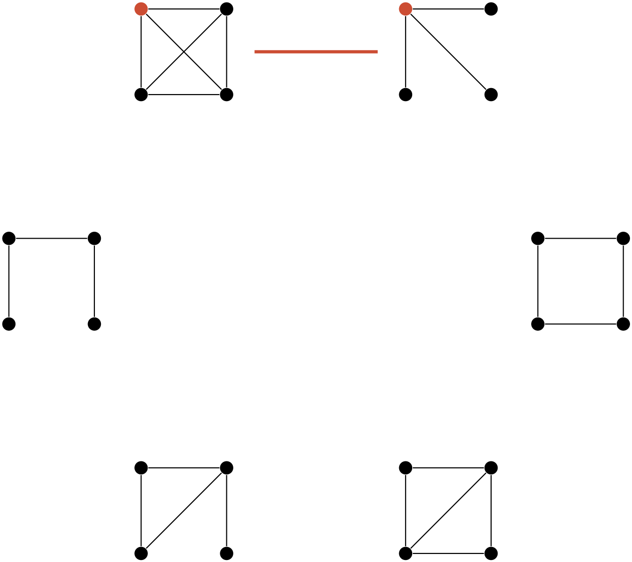

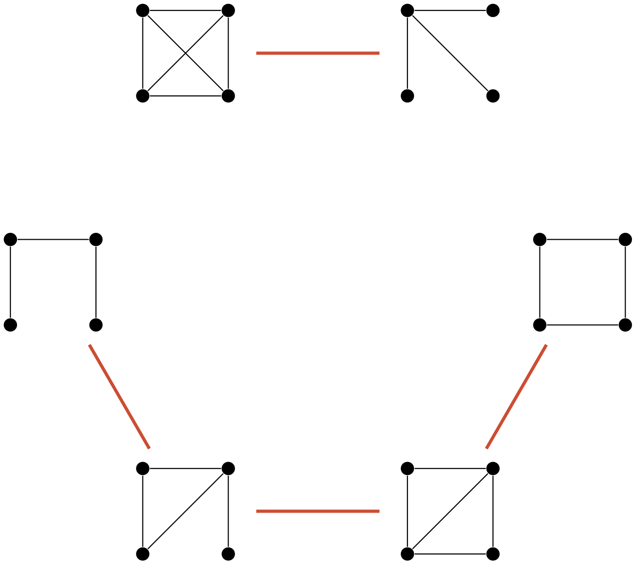

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How many equivalence classes does $\sim_C$ have?

Strategies:

- Graph of graphs

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- Sorting by the number of edges

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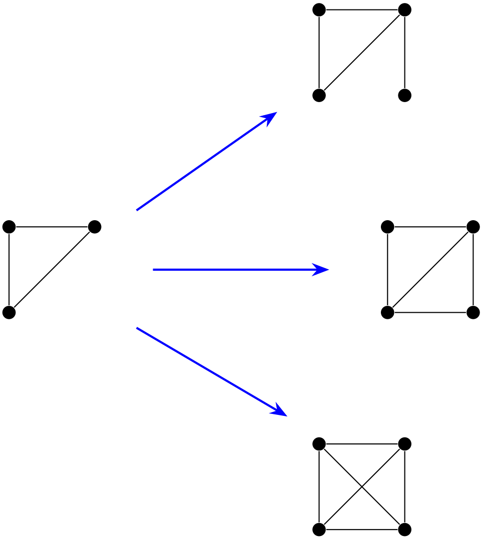
Strategies:

- Graph of graphs
- Sorting by the number of edges
- Recursion

Graphs



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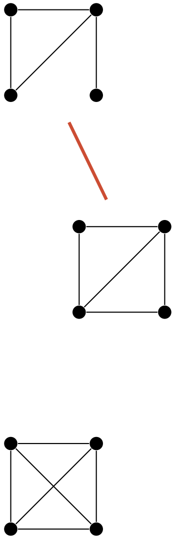

Question

How many equivalence classes does $\sim_C$ have?

$p \backslash n$	1	2	3	4	5	6	7	8	9	$\cdots$	12
2	1	1	1	2	4	11	26	101	440	$\cdots$	1274068
3	1	1	1	3	5	21	73	?	?	$\cdots$	?

Stabiliser codes

$\mathcal{H} := (\mathbb{C}^p)^{\otimes n}$

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Quantum error-correcting code

An $((n, k, d))_p$ quantum error-correcting code is a k-dimensional subspace of $\mathcal{H}$ for which all errors of weight at most $d - 1$ can be detected.

Stabiliser codes

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Stabiliser code

$$\mathcal{C} = \{\psi \in \mathcal{H} \mid S\psi = \psi \text{ for all } S \in \mathcal{S}\} \quad \text{where } \mathcal{S} \leqslant \mathcal{P}_n$$

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$\mathcal{P}_n$ is the **Pauli group**

$$\boxed{p = 2}$$

Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad and \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli group

$$\begin{aligned} \mathcal{P} =& \langle X, Y, Z \rangle \\ =& \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\} \end{aligned}$$

$$\omega^p = 1, \omega \neq 1, \zeta^2 = \omega$$

Shift operator

$$X = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Clock operator

$$Z = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \omega & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \omega^{p-2} & 0 \\ 0 & 0 & \cdots & 0 & \omega^{p-1} \end{pmatrix}$$

Stabiliser codes

$\omega^p = 1, \omega \neq 1, \zeta^2 = \omega$

Pauli group

$$\mathcal{P}_n := \left\{ \zeta^\lambda X^{\vec{x}} Z^{\vec{z}} \mid \lambda = 1, \ldots, 2p \text{ and } \vec{x}, \vec{z} \in (\mathbb{F}_p)^n \right\}$$

where $X^{\vec{x}} Z^{\vec{z}} := X^{x_1} Z^{z_1} \otimes \cdots \otimes X^{x_n} Z^{z_n}$

Stabiliser codes

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$$ZX = \omega XZ$$

Stabiliser code

$\mathcal{C} = \{\psi \in \mathcal{H} \mid S\psi = \psi$ for all $S \in \mathcal{S}\}$ where $\mathcal{S} \leqslant \mathcal{P}_n$

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Stabiliser code

$\mathcal{C} = \{\psi \in \mathcal{H} \mid S\psi = \psi \text{ for all } S \in \mathcal{S}\}$ where $\mathcal{S} \leqslant \mathcal{P}_n$

Stabiliser group

$\mathcal{S} = \{P \in \mathcal{P}_n \mid P\psi = \psi \text{ for all } \psi \in \mathcal{C}\}$

Stabiliser codes

$$\mathcal{S} = \left\langle \zeta^{\lambda_i} X^{\vec{x}_i} Z^{\vec{z}_i} \right\rangle_{1 \leq i \leq n-k}$$

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$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-k} \end{pmatrix} \quad \text{and} \quad G = \left(\begin{array}{ccc|ccc} x_{11} & \cdots & x_{1n} & z_{11} & \cdots & z_{1n} \\ x_{21} & \cdots & x_{2n} & z_{21} & \cdots & z_{2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n-k,1} & \cdots & x_{n-k,n} & z_{n-k,1} & \cdots & z_{n-k,n} \end{array}\right)$$

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Example ($p = 3$, $n = 3$)

$$\begin{aligned} \mathcal{S} = \langle XZ^2 \otimes Z \otimes I,& \\ X \otimes XZ \otimes Z\rangle& \end{aligned} \implies G = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right)$$

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Clifford group

$$\mathcal{C}_n := \left\{ U \in \left(\mathbb{C}^{p\times p}\right)^{\otimes n} \mid UU^\dagger = I \text{ and } U^\dagger P U \in \mathcal{P}_n \text{ for all } P \in \mathcal{P}_n \right\}$$

Clifford group equivalence

$\mathcal{C} \sim_C \mathcal{C}' \iff \mathcal{S}' = U^\dagger \mathcal{S} U$ for some $U \in \mathcal{C}_n$,
modulo a permutation of the factors in the tensor product

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Example

X and Z are Clifford operators

From now on, let $k = 0$.

$$G = \left(\begin{array}{ccc|ccc} x_{11} & \cdots & x_{1n} & z_{11} & \cdots & z_{1n} \\ x_{21} & \cdots & x_{2n} & z_{21} & \cdots & z_{2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} & z_{n1} & \cdots & z_{nn} \end{array}\right)$$

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Theorem

$\mathcal{C}$ is equivalent to a code where G is of the form $(I \mid A)$, with A an adjacency matrix of a weighted graph.

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$$G = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & a_{12} & \cdots & a_{1n} \\ 0 & 1 & \cdots & 0 & a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n1} & a_{n2} & \cdots & 0 \end{array}\right)$$

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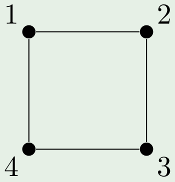
$\rightarrow$ *graph state*

$$G = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & a_{12} & \cdots & a_{1n} \\ 0 & 1 & \cdots & 0 & a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n1} & a_{n2} & \cdots & 0 \end{array}\right)$$

Example ($p = 2$)



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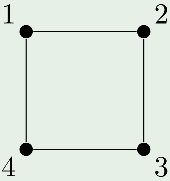

$$G = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right)$$

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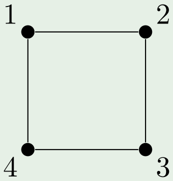

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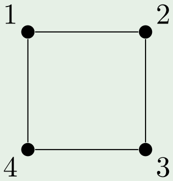

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$\mathcal{C}$ is a $((4,1,2))_2$ code

Stabiliser codes

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Theorem

Let $\mathcal{C}$ and $\mathcal{C}'$ be one-dimensional stabiliser codes with stabiliser groups $\mathcal{S}$ and $\mathcal{S}'$ and graphs Γ and Γ' respectively. Then

$$\mathcal{C} \sim_C \mathcal{C}' \iff \Gamma \sim_C \Gamma'$$

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Question

How many equivalence classes does $\sim_C$ have?

Strategies:

- Graph of graphs
- Sorting by the number of edges
- Recursion
- Explicit Clifford operator

Thank you for listening!

Finite geometry

$$G = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & a_{12} & \cdots & a_{1n} \\ 0 & 1 & \cdots & 0 & a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n1} & a_{n2} & \cdots & 0 \end{array}\right)$$

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$\rightarrow$ lines in $\mathrm{PG}(n-1, p)$

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$\rightarrow$ lines in $\mathrm{PG}(n-1,p)$

$\rightarrow$ **quantum set of lines**

Theorem

$\mathcal{X}$ is a quantum set of lines of $\mathrm{PG}(n-k-1,2)$ iff every codimension 2 subspace is skew to an even number of the lines in $\mathcal{X}$.